Stability Analysis of Closed-loop Systems in the Z-domain

\[ \frac{C(z)}{R(z)} = \frac{G(z)}{1 + GH(z)} \]

- The stability of a closed-loop system may be determined from the locations of the closed-loop poles in the z-plane or the roots of the characteristic equation:

\[ P(z) = 1 + GH(z) = 0 \]

as follows:

1. The closed-loop poles or the roots of the characteristic equation must lie within the unit circle in the z-plane. Any closed-loop pole outside the unit circle makes the system unstable.

2. If a single pole lies at \( z = 1 \) or \( z = -1 \) (or a complex pole lies at \( z = 1 \) and another complex pole at \( z = -1 \)), then the system becomes marginally (critically) stable. Also, the system becomes critically stable if a single pair of complex conjugate poles lie on the unit circle in the z-plane. Any multiple closed-loop pole on the unit circle makes the system unstable.

3. Closed-loop zeros do not affect the absolute stability and therefore may be located anywhere in the z-plane.
Example

Consider the closed-loop control system shown in the figure below. Determine the stability of the system when \( K = 1 \).

\[
G(s) = \frac{1 - e^{-st}}{s} \frac{K}{S(s+1)}
\]

\[
\therefore G(z) = \left(1 - \frac{1}{z}\right) \left[ \frac{(368z + 264)}{(z-1)(z-368)} \right]
\]

\[
= \frac{368z + 264}{(z-1)(z-368)}
\]

\[
\frac{C(z)}{R(z)} = \frac{G(z)}{1 + G(z)}
\]

C.E:

\[
1 + G(z) = 0
\]

\[
(z-1)(z-368) + 368z + 264 = 0
\]

\[
z^2 - z + 0.632 = 0
\]

\[
(z - 0.5 \pm 0.618) = 0
\]

\[
\therefore z_1 = 0.5 + 0.618, \quad z_2 = 0.5 - 0.618
\]

Since \(|z_1| = |z_2| < 1\),

the system is stable.
Methods for testing absolute stability

Bilinear Transformation and Routh Stability criterion

Bilinear Transformation

- Many analysis and design techniques for continuous-time systems, such as the Routh-Hurwitz criterion and Bode techniques, are based on the property that in the S-plane the stability boundary is the imaginary axis.

- Thus, these techniques cannot be applied to discrete-time systems in the Z-plane, since the stability boundary is the unit circle.

- However, through the use of transformation

\[ Z = \frac{1 + WT/2}{1 - WT/2} \]

or

\[ W = \frac{2}{T} \frac{Z-1}{Z+1} \]

the unit circle of the Z-plane transforms into the imaginary axis of the W-plane.

- This can be seen through the following development

\[ W = \frac{2}{T} \frac{Z-1}{Z+1} \bigg|_{Z = e^{j\omega T}} \]

\[ = \frac{2}{T} \frac{e^{j\omega T} - 1}{e^{j\omega T} + 1} = \frac{2}{T} \frac{e^{j\omega T/2} - e^{-j\omega T/2}}{e^{j\omega T/2} + e^{-j\omega T/2}} \]

\[ = \frac{2}{T} \tan \frac{WT}{2} \]

- Thus, it is seen that the unit circle of the Z-plane transforms into the imaginary axis of the W-plane.
For small values of real frequencies (s-plane frequency), such that \( W_T \) is small,

\[ W_W = \frac{2}{1} \tan \frac{W_T}{2} = W_T \]

Thus the \( W \)-plane frequency is approximately equal to the s-plane frequency for this case.

This approximation is valid for those values of frequency for which

\[ \tan \left( \frac{W_T}{2} \right) \approx \frac{W_T}{2} \]

or

\[ \frac{W_T}{2} \leq \frac{\pi}{10} \]

\[ W \leq \frac{2\pi}{10\pi} = \frac{W_S}{10} \]

The error in this approximation is less than 4%.

Because of the phase lag introduced by the ZOH, we usually choose the sample period \( T \) such that \( W \leq \frac{W_S}{10} \) is satisfied over most of not all of the band of frequencies that the system will pass (the system bandwidth).

At \( W = \frac{W_S}{10} \), the ZOH introduces a phase lag of 18°, which is an appreciable amount.
Mapping from $s$-plane to $z$-plane to $w$-plane

$$W = \int W_w = j T \tan \frac{\omega T}{2}$$

$$w_w = \frac{2}{T} \tan \frac{\omega T}{2}$$
The Routh-Hurwitz Criterion

- The Routh-Hurwitz criterion may be used in the analysis of continuous-time systems to determine if any roots of a given equation are in the right half of the s-plane.

- If the criterion is applied to the characteristic equation of a discrete-time system when expressed as a function of z, no useful information on stability is obtained.

- However, if the characteristic equation is expressed as a function of the bilinear transform variable w, then the stability of the system may be determined by directly applying the R-H criterion.

Example: Consider the system

\[ G(s) = \frac{\frac{7}{3}}{s(s+1)} \]

Determine the values of K for stability.

\[ G(s) = \frac{1-e^{-st}}{s} \cdot \frac{K}{s(s+1)} \]

\[ G(z) = \frac{0.0484z + 0.0468}{(z-1)(z-0.95)} \]

Then \( G(w) \) is given by

\[ G(w) = G(z) \left| z = \frac{1+wT/2}{1-wT/2} = \frac{1+0.05w}{1-0.05w} \right. \]

\[ = K \frac{-0.0016w^2 - 1.872w + 3.81}{3.81w^2 + 3.80w} \]
C.E. is

\[ 1 + g(w) = (3.81 - 0.00016k)w^2 + (3.8 - 1.872k)w + 3.81k = 0 \]

\[
\begin{array}{c|c}
W^2 & 3.81-0.00016k \\
W & 3.8-1.872k \\
W^0 & 3.81k \\
\end{array}
\]

\[ 3.81k \Rightarrow k < 23.81 \]
\[ 3.8 \Rightarrow k < 20.3 \]
\[ 0 \Rightarrow k > 0 \]

Hence, for no sign change to occur in the first column, it is necessary that \( k \) be in the range \( 0 < k < 20.3 \) and this is the range of \( k \) for stability.

The frequency of oscillation at \( k = 20.3 \) can be found from the \( W^2 \) row of the array

\[ (3.81-0.00016k)w^2 + 3.81k \]

\[ k = 20.3 \]

\[ 3.807w^2 + 77.343 = 0 \]

\[ w_w = 4.5 \]

\[ w = \frac{2}{T} \tan^{-1} \frac{w_w}{2} = 4.934 \text{ rad/s} \]

and is the s-plane (real) frequency at which the system will oscillate if \( k \) is set equal to 20.3.
Example: For the previous example, let \( T = 1 \text{sec} \). Find the value of \( K \) for stability.

Set:

Going through the same step,

\[ 0 < K < 2.39 \text{ for stability} \]

and \( \omega_w = 1.559 \)

\[ \omega = \frac{2}{T} \tan^{-1} \frac{\omega_w}{2} \]

\[ = 1.32 \text{ rad/s} \]

- The degradation of system stability as \( T \) increases from 0.1 to 1 sec is clear from the example.

- This is due to the delay introduced by the sampler and data hold, which was illustrated in the frequency response of the data hold circuit.
The Jury Stability Test:

- May be applied to polynomial equations with real coefficients. Since the coefficients of the characteristic equations corresponding to physically realizable systems are always real, Jury test can be applied.

- In order to apply the Jury stability test to a given characteristic equations $P(z)=0$, we construct a table whose elements are based on the coefficient of $P(z)$. Assume the C.E. $P(z)$ as:

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

Then, the general form of the Jury table becomes:

<table>
<thead>
<tr>
<th>Row</th>
<th>$z^0$</th>
<th>$z^1$</th>
<th>$z^2$</th>
<th>\ldots</th>
<th>$z^n$</th>
<th>$z^{n-1}$</th>
<th>$z^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$a_n$</td>
<td>$a_{n-1}$</td>
<td>$a_{n-2}$</td>
<td>\ldots</td>
<td>$a_2$</td>
<td>$a_1$</td>
<td>$a_0$</td>
</tr>
<tr>
<td>2</td>
<td>$a_0$</td>
<td>$a_1$</td>
<td>$a_2$</td>
<td>\ldots</td>
<td>$a_{n-2}$</td>
<td>$a_{n-1}$</td>
<td>$a_n$</td>
</tr>
<tr>
<td>3</td>
<td>$b_{n-1}$</td>
<td>$b_{n-2}$</td>
<td>$b_{n-3}$</td>
<td>\ldots</td>
<td>$b_1$</td>
<td>$b_0$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$b_0$</td>
<td>$b_1$</td>
<td>$b_2$</td>
<td>\ldots</td>
<td>$b_{n-2}$</td>
<td>$b_{n-1}$</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$c_{n-2}$</td>
<td>$c_{n-3}$</td>
<td>$c_{n-4}$</td>
<td>\ldots</td>
<td>$c_0$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$c_0$</td>
<td>$c_1$</td>
<td>$c_2$</td>
<td>\ldots</td>
<td>$c_{n-2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>\vdots</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
where
\[ b_k = \begin{bmatrix} a_n & a_{n-k} \\ a_0 & a_{k+1} \end{bmatrix} \quad ; \quad k = 0, 1, 2, \ldots, n-1 \]
\[ c_k = \begin{bmatrix} b_{n-1} & b_{n-2-k} \\ b_0 & b_{k+1} \end{bmatrix} \quad ; \quad k = 0, 1, 2, \ldots, n-2 \]
\[ q_k = \begin{bmatrix} P_3 & P_{2-k} \\ P_0 & P_{k+1} \end{bmatrix} \quad ; \quad k = 0, 1, 2 \]

Note that the last row in the table consists of three elements. For second-order systems, \(a_{n-3} = 4 - 3 = 1\) and the Jury table consists only of one row containing three elements.

**Stability Criterion by the Jury test:**

A system with the characteristic equation \(P(z) = 0\) is stable if the following conditions are all satisfied:

(i) \(|a_n| < |p_0|\)
(ii) \(P(1) > 0\)
(iii) \(P(-1) = \begin{cases} > 0 & \text{for } n \text{ even} \\ < 0 & \text{for } n \text{ odd} \end{cases}\)
(iv) \(|b_{n-1}| > |b_0|\)
\(|c_{n-2}| > |c_0|\)
\(|q_2| > q_0|\)
**Example:** Consider the system

\[ G(z) = \begin{cases} 0.00484 z + 0.00468 \\ (z-1)(z-0.905) \end{cases} \]

Determine the value of K for stability using the Jury stability method.

**SOL:**

\[ G(z) = K \frac{0.00484 z + 0.00468}{(z-1)(z-0.905)} \]

**C.E.:**

\[ P(z) = 1 + G(z) = 1 + K \frac{0.00484 z + 0.00468}{(z-1)(z-0.905)} = 0 \]

\[ z^2 + (0.00484K-1.905)z + (0.905+0.00468K) = 0 \]

The Jury array is

\[
\begin{array}{ccc}
 z^0 & z^1 & z^2 \\
0.00468K+0.905 & 0.00484K & -1.905 \\
1 & & \\
\end{array}
\]

The constraint

\[ |a_2| < |a_0| \text{ yields} \]

\[ 0.00468K+0.905 < 1 \]

\[ 0 < K < 0.3 \]

The constraint \( P(1) > 0 \) yields

\[ 1 + 0.00484K - 1.905 + 0.905 + 0.00468K > 0 \]

\[ K > 0 \]

The constraint \( P(-1) > 0 \) yields

\[ 1 - 0.00484K + 1.905 + 0.905 + 0.00468K > 0 \]

\[ K < 23.813 \]

**The system is stable for**

\[ 0 < K < 20.3 \]
The system is marginally stable for $k = 20.3$. For this value of $k$, the C.E. is

$$Z^2 + 1.806748Z + 1.000004 = 0$$

The roots of the equation are

$$Z = -0.9034 \pm j \cdot 0.429 = 1 \angle 25.4^0 = 1 \angle \pm 44.34 \text{ rad}$$

$$= 1 \angle \pm \omega_T$$

Since $T = 0.1$, the system will oscillate at a frequency of $4.434 \text{ rad/s}$. The results match with the results obtained using Bilinear transformation and R-H test.