Gentle Introduction to First-Order Logic Model Theory for Knowledge Representers

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AIM: TO REPRESENT THE STATE OF THE WORLD USING SYMBOLS FOR COMMUNICATING OUR KNOWLEDGE TO THE COMPUTER.

E.g. Represent

SOCRATES IS A MAN.

ALL MEN ARE MORTALS.

Infer

SOCRATES IS A MAN.

SOME MEN ARE MORTAL.

SOCRATES IS MORTAL.
To represent knowledge, we need a suitable language.

To specify what valid conclusions can be inferred from a specification, we need logic.

Logic

Language (syntax)

Models (semantics)

Deduction (pragmatics)
Task of a Knowledge Representor

A relevant information about express the state of the world of interest in the language of logic.

Task of the Computer (Reasoning Program)

Infer the intent of the representor implicit in the representation

(* disciplined fill-in gaps in what is explicitly specified *)
Example

Information about class

m( Venu )
f( mary )
f( mary ellen )
f( barbara )

What possible states of the world can I have in my mind?

Required to reconstruct our world by the intelligent program.
DOMAIN = \{ \}

MALE = \{ \}

FEMALE = \{ \}

m \rightarrow \text{MALE}

f \rightarrow \text{FEMALE}

venu \rightarrow \}

mary \rightarrow \}

mary ellen \rightarrow \}

barbara \rightarrow \}
DOMAIN = { GLENN, CLOSE, M, L

MALE = { GLENN, CLOSE

FEMALE = { GLENN, CLOSE

\& GLENN \textit{\textbf{RELEVANT}} \textit{\textbf{FOR}} \textit{\textbf{REPRESENTATION}}

DOMAIN = { GLENN, SEABORG, M, L

MALE = { GLENN, SEABORG

FEMALE = { GLENN, SEABORG

\textit{\textbf{BOTH THESE STATES SATISFY OUR FACTS.}}
OTHER POSSIBLE MODELS

\[
\text{DOMAIN } = \{ \text{men, women, Mary} \}
\]

\[
\text{m } \mapsto \text{CS.TEACHER } = \{ \text{man} \}
\]

\[
\text{f } \mapsto \text{CS.STUDENTS } = \{ \text{woman} \}
\]

\[
\text{m } \mapsto \text{INDIAN.CITIZEN } = \{ \text{man} \}
\]

\[
\text{f } \mapsto \text{US.CITIZEN } = \{ \text{woman} \}
\]

\[
\text{m } \mapsto \text{spectacled } = \{ \text{man} \}
\]

\[
\text{f } \mapsto \text{non-spectacled } = \{ \text{woman} \}
\]
**World of Earthworms**

**Domain:** \( \{ m^m, m^b, m^m \} \)

\[ M \rightarrow \text{Male}_E = \{ m^m, m^b, m^m \} \]

\[ f \rightarrow \text{Female}_E = \{ m^m, m^b, m^m \} \]

*Earthworms are Hermaphrodite.*

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**World of Amoebas**

**Domain:** \( \{ y, b, w \} \)

\[ m \rightarrow \text{Male}_A = \emptyset \]

\[ f \rightarrow \text{Female}_A = \emptyset \]

*Amoebas are asexual.*
IN ALL THE "MODELS" WE HAVE GIVEN

\[ m(venu) \] and

\[ f(mary) \]

ARE ALWAYS TRUE.

\[
\text{/* THEOREM */}
\]

\[ m(glenn) \] and \[ f(glenn) \]

are true in some "models" and false in others.

\[
\text{/* NON-THEOREMS */}
\]
UNINTENDED YET A POSSIBLE MODEL

\[ \text{DOMAIN} = \{ \text{man, woman, child} \} \]

\[ m \mapsto \{ \text{man, woman, child} \} \]

\[ f \mapsto \{ \text{man, woman, child} \} \]

To rule out this state of the world we add

\[ \forall x : m(x) \Rightarrow \exists f(x) \]

\[ \forall x : f(x) \Rightarrow \exists m(x) \]

The above interpretation is no longer a model.
Abstract Model

A Model of Sex Symbols.

\[ m(v), f(b), f(m). \]

\[ \forall x m(x) \iff \neg f(x) \]

Is satisfied by

**Domain = \{ A, \ldots, \emptyset, \alpha, \beta, \ldots, \emptyset \}**

\[ v \mapsto \emptyset \quad b \mapsto \emptyset \quad m \mapsto \emptyset \]

\[ m \mapsto \{ A, \ldots, \emptyset \} \quad f \mapsto \{ \emptyset, \ldots, \emptyset \} \]

But is not satisfied if

**Domain = \{ A, \ldots, \emptyset, \ldots, \emptyset \}**

\[ m \mapsto \{ A, \ldots, \emptyset \} \quad f \mapsto \{ \emptyset, \ldots, \emptyset \} \]
To pin down a domain as computer can guess only based on syntax we introduce Herbrand domain and Herbrand models.

\[
\begin{align*}
\{ m(x), f(c) \} \\
\forall x : m(x) \iff f(x) \\
\forall x : m(x) \lor f(x)
\end{align*}
\]

Domain = \{ "a", "b" \}

\[
\begin{align*}
& a \mapsto \text{"a"} \\
& b \mapsto \text{"b"}
\end{align*}
\]

\[
\begin{align*}
& m \mapsto \{ \text{"a"} \} \\
& f \mapsto \{ \text{"b"} \}
\end{align*}
\]
KNOWLEDGE
REPRESENTATION
CONSISTS OF SPECIFYING
OBJECTS IN THE WORLD
AND
RELATIONSHIPS AMONG
THEM.
FORMALIZATION OF CONNECTION BETWEEN SYMBOLS IN A FIRST-ORDER LANGUAGE AND THEIR MEANING

- A logic language has
  - constants
  - function symbols

which form TERMS to represent objects

- predicate symbols to represent relations
To interpret the language we fix a domain of discourse $D$ and map
- constants to individuals in domain $D$.
- function symbols to functions over domain $D$.
- predicate symbols to relations over domain $D$. 
SENTENCES IN A FIRST-ORDER LOGIC LANGUAGE ARE FORMED USING THE FOLLOWING GRAMMAR.

\[
\text{<term>} ::= \text{constant} \\
\quad | \text{variable} \\
\quad | \text{func-sym(}<\text{term}>\text{)}
\]

\[
\text{<sentence>} ::= \text{pred-sym(}<\text{term}>\text{)} \\
\quad | \text{<sentence>} \text{^} \text{<sentence>} \\
\quad | \text{<sentence>} \text{v} \text{<sentence>} \\
\quad | \text{I} \text{<sentence>} \\
\quad | \text{T} \text{<sentence>} \\
\quad | \text{variable} \text{<sentence>} \\
\quad | \text{\exists variable} \text{<sentence>}
\]
A PEEK AT THE DEFINITION
OF SATISFACTION

An interpretation $I$ satisfies a sentence $\phi$:

$I \models \text{pred}_\text{sym}(\text{term})$

iff $I(\text{term}) \in I(\text{pred}_\text{sym})$

$I \models \forall x : P_\text{Sym}_1(x) \Rightarrow P_\text{Sym}_2(x)$

iff for all $d \in D$

whenever $I \models P_\text{Sym}_1(d)$

it is the case that

$I \models P_\text{Sym}_2(d)$. 
A model of a sentence is an interpretation of the sentence that satisfies the "constraints" imposed by the sentence. A theorem is a sentence that is true in all models of the sentence.
More Examples

\[ \text{number } (\text{zero}) \]
\[ \text{number } (\text{succ } (X)) \]
\[ := \text{number } (X). \]

\underline{Model 1}

\[ D = \{0, 1, 2, \ldots 3\} \]
\[ I(\text{zero}) = \emptyset \]
\[ I(\text{succ}) = \lambda x . 1+x \]
\[ I(\text{number}) = D \]

\underline{Model 2}

\[ D = \text{LISP s-expressions} \]
\[ I(\text{zero}) = \text{nil} \]
\[ I(\text{succ}) = \lambda s . \text{cons } (s, \text{nil}) \]
\[ I(\text{number}) = \{ \text{nil}, \text{cons } (\text{nil, nil}) \ldots 3 \} \]
Model 3

\[ D = \{ 1, 3 \} \]

\[ I(\text{zero}) = 1 \]
\[ I(\text{succ}) = \forall x \cdot 1 \]
\[ I(\text{number}) = \{ 1, 3 \} \]

Model 4

A Herbrand Model

Domain = \{ "zero", \text{succ}(\text{zero}), \text{succ}(\text{succ}(\text{zero})), \ldots \} \]

\[ I(\text{zero}) = \text{zero} \]
\[ I(\text{succ}) = \forall x \cdot \text{succ}(x) \]
\[ I(\text{number}) = \{ \text{zero}, \text{succ}(\text{zero}), \text{succ}(\text{succ}(\text{zero})), \ldots \} \]
Herbrand Model

* Domain is composed of all variable-free terms that can be formed using constants and function symbols of the language.

* No two different terms can be mapped to the same entity by the interpretation.

\[ \text{UNIQUE NAME} \]
Meaning of a Logic Program

Set of all Herbrand Models

(\textit{Theorems are positive and belong to all the models \#})

(\textit{This characterization is declarative, as opposed to operational \#})

Meaning of logic programs when negation is allowed in the query

Intersection of all Herbrand Models = Minimum Model
Response to Variable-Free Query

YES: Query true in all Herbrand Models

NO: Query not true in all Herbrand Models

Alternatively,

YES if and only if true in minimal Herbrand model
Prolog Computation

Deduction in Classical Logic

Incompleteness of Prolog Interpreter

\[ p \iff \neg p \]

\[ q \iff q \]

\[ p \text{ is not a theorem, but Prolog goes into } \infty \text{-loop} \]

\[ q \text{ is a theorem, yet Prolog goes into an } \infty \text{-loop} \]
COMPUTATION AS DEDUCTION

(First-order logic)

Theorem proving

Sentences \(\rightarrow\) Models \(\rightarrow\) Theorems

Common-sense reasoning

Sentences + Minimality Constraints \(\rightarrow\) "MINIMAL" Models \(\rightarrow\) Conclusions